On $s\pi$-Weakly Regular Rings

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ABSTRACT

A ring $R$ is said to be right(left) $s\pi$-weakly regular if for each $a \in R$ and a positive integer $n$, $a^n \in a^n R a^{2n} R (a^n \in R a^{2n} R a^n)$. In this paper, we continue to study $s\pi$-weakly regular rings due to R. D. Mahmood and A. M. Abdul-Jabbar [8]. We first consider properties and basic extensions of $s\pi$-weakly regular rings, and we give the connection of $s\pi$-weakly regular, semi $\pi$-regular and $\pi$-biregular rings.

Key words: weakly regular rings , reduced rings , $\pi$-biregular rings.

1. Introduction

Throughout this paper rings are associative with identity. For a subset $X$ of a ring $R$, the right annihilator of $X$ in $R$ is $r(X) = \{ r \in R : xr = 0, \text{ for all } x \in X \}$. $J( R )$, $Y( R )$ and $N$ will stand respectively for the Jacobson radical, right singular ideal of $R$ and the set of all nilpotent elements. A ring
R is called right (left) \( s \)-weakly regular if for each \( a \in R \), \( a \in aRa^2 R (a \in Ra^2 Ra) \). This concept was introduced by V. Gupta [4] and W. B. Vasantha Kandasamy [10]. As a generalization of this concept the authors in [8] defined \( \pi \)-weakly regular ring that is a ring such that for each \( a \in R \) and a positive integer \( n \), \( a^n \in a^aR a^{2n} R (a^n \in R a^{2n} R a^n) \). In the present work we develop further properties of \( \pi \)-weakly regular rings, and we give the connection of \( \pi \)-weakly regular rings with other rings. Recall that:

1. \( R \) is called reduced if it has no nonzero nilpotent elements.
2. According to Cohn [3], a ring \( R \) is called reversible if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \). Obviously every reduced ring is reversible. It is easy to see that \( R \) is reversible if and only if right (left) annihilator in \( R \) is a two-sided ideal.

2. \( s \pi \)WR-Rings

In this section we continue to study \( s \pi \)WR-rings and we establish relation between such rings, semi \( \pi \)-regular and \( \pi \)-biregular rings.

Definition 2.1[8]:

\( R \) is called right(left) \( s \pi \)WR-ring if for each \( a \in R \), there exists a positive integer \( n = n(a) \), depending on \( a \) such that \( a^n \in a^aR a^{2n} R (a^n \in R a^{2n} R a^n) \). \( R \) is called \( s \pi \)WR-ring if it is both right and left \( s \pi \)WR-ring.

Recall that, a right ideal \( P \) of a ring \( R \) is said to be completely prime if for all \( a, b \in R \) such that \( ab \in P \), then either \( a \in P \) or \( b \in P \).

Proposition 2.2:

If \( R \) is a right \( s \pi \)WR-ring and \( r(a^n) \subset r(a) \), for every \( a \in R \) and a positive integer \( n \). Then every completely prime ideal of \( R \) is maximal.

Proof:

Let \( P \) be a completely prime ideal of \( R \) and \( J \) any ideal of \( R \) such that \( P \subset J \). Then, there exists \( x \in J \) and \( x \notin P \). Since \( R \) is right \( s \pi \)WR-ring, there exist \( b, c \in R \) and a positive integer \( n \) such that \( x^n = x^n b x^{2n} c \) and \( x(1-b x^{2n} c) = 0 \) (since \( r(a^n) \subset r(a) \)). So, \( x(1-b x^{2n} c) = 0 \in P \). Since \( x \notin P \) and \( P \) is completely prime ideal of \( R \), so \( (1- b x^{2n} c) \in P \subset J \), and hence \( (1- b x^{2n} c) \in J \). Hence \( 1- b x^{2n} c + b x^{2n} c = 1 \in J \) and therefore, \( J = R \). Consequently \( P \) is a maximal ideal of \( R \). ♦

Theorem 2.3:

If \( R \) is a right \( s \pi \)WR-ring and \( I \) is an ideal of \( R \), then \( R/I \) is \( s \pi \)WR-ring.
Proof:
Let R be a right $\pi\WR$-ring. Then, there exist $b, c \in R$ such that $a^n = a^n b a^{2n} c$, for some positive integer $n$. Now,

$$(a + I)^n (b + I) (a + I)^{2n} (c + I) = (a^n + I) (b + I) (a^{2n} + I) (c + I)$$

$$= a^n b a^{2n} c + I$$

$$= a^n + I.$$ 

Therefore, $R/I$ is a right $\pi\WR$-ring. ♦

Now, the following result is given in [7] and [1].

**Lemma 2.4:**

Let R be a reduced ring. Then, for every $a \in R$ and every positive integer $n$,

1. $r(a^n) = \ell (a^n)$.
2. $a^n R \cap r(a^n) = 0$.
3. $r(a) = r(a^n)$.

**Theorem 2.5:**

Let $R$ be a reduced ring. Then, $R$ is a right $\pi\WR$-ring if and only if $R / r(a^m)$ is $\pi\WR$-ring, for every $a \in R$ and a positive integer $m$.

**Proof:**

Assume that $R / r(a^m)$ is a right $\pi\WR$-ring, for every $a \in R$ and a positive integer $m$. Now,

$$a^n + r(a^m) = (a^n + r(a^m)) (b + r(a^m)) (a^{2n} + r(a^m)) (c + r(a^m)),$$

$$= a^n b a^{2n} c + r(a^m).$$

Then, $a^n - a^n b a^{2n} c \in r(a^m)$, this implies that $a^m (a^n - a^n b a^{2n} c) = 0$ and hence $1 - b a^{2n} c \in r(a^{m+n}) = r(a^n)$ since $R$ is reduced. So, $a^n (1 - b a^{2n} c) = 0$ and $a^n = a^n b a^{2n} c$. Therefore, $R$ is a right $\pi\WR$-ring.

Conversely, assume that $R$ is a right $\pi\WR$-ring. Then, by Theorem 2.3, $R / r(a^m)$ is an $\pi\WR$-ring. ♦

Next we consider the Jacobson radical, the right singular ideal of $a$ and the set of all nilpotent elements $N$ of right $\pi\WR$-ring.

**Theorem 2.6:**

Let $R$ be a right $\pi\WR$-ring. Then,

1. $J( R ) = N$
2. If $R$ is a reduced, then $Y( R )$ is a nilideal of $R$.  

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Proof:

(1) Let $0 \neq x \in J(\mathcal{R})$. Then, there exist $b, c \in \mathcal{R}$ and a positive integer $n$ such that $x^n = x^n b x^{2n} c$. Then, $1 - b x^{2n} c$ is invertible. Therefore, there exists an invertible element $u$ such that $(1 - b x^{2n} c) u = 1$. Multiply from the left by $x^n$, we obtain $(x^n - x^n b x^{2n} c) u = x^n$. Whence it follows that $x^n = 0$, so $x \in \mathcal{N}$ and $J(\mathcal{R}) \subseteq \mathcal{N}$. Since $\mathcal{N} \subseteq J(\mathcal{R})$. Thus, $J(\mathcal{R}) = \mathcal{N}$.

(2) Let $a$ be a nonzero element in $Y(\mathcal{R})$. Then, $r(a)$ is an essential right ideal of $\mathcal{R}$. Since $\mathcal{R}$ is right $\pi$-WR-ring, there exist $b, c \in \mathcal{R}$ and a positive integer $n$ such that $a^n = a^n b a^{2n} c$. Consider $r(a^n) \cap b a^{2n} R$, let $x \in r(a^n) \cap b a^{2n} R$. Then, $a^n x = 0$ and $x = b a^{2n} t$, for some $t \in \mathcal{R}$. So, $a^n b a^{2n} t = 0$, then $a^n = 0$. Thus, $a^{2n} = 0$ and hence $b a^{2n} t = 0$, yields $x = 0$. Therefore, $r(a^n) \cap b a^{2n} R = 0$. Since $r(a^n)$ is a nonzero essential right ideal of $\mathcal{R}$, then $b a^{2n} = 0$ and hence $a^{2n} = 0$. Thus, $a^{n} = 0$. So, $Y(\mathcal{R})$ is a nilideal of $\mathcal{R}$.

Recall that, a ring $\mathcal{R}$ is called right(left) semi $\pi$-regular [1] if for every $a \in \mathcal{R}$, there exists $b \in \mathcal{R}$ and a positive integer $n$ such that $a^n = a^n b$ and $r(a^n) = r(b) (a^n = b a^n$ and $\ell (a^n) = \ell (b))$.

The following theorem gives the conditions of $s\pi$WR-ring to be semi $\pi$-regular.

**Theorem 2.7:**
If $\mathcal{R}$ is a reversible, $s\pi$WR-ring and $a^{2n} \mathcal{R} = \mathcal{R} a^{2n}$, for every $a \in \mathcal{R}$ and a positive integer $n$, then $\mathcal{R}$ is semi $\pi$-regular.

**Proof:**

Let $\mathcal{R}$ be $s\pi$WR-ring. Then, for every $a \in \mathcal{R}$, there exist $b, c \in \mathcal{R}$ and a positive integer $n$ such that $a^n = a^n b a^{2n} c$. Since $b a^{2n} \mathcal{R} a^{2n} = a^{2n} \mathcal{R}$. So, $a^n = a^{3n} h c$, for some $h \in \mathcal{R}$. This implies that $a^n = a^n a^{2n} t$ (set $t = hc$). If we set $y = a^{2n} t$, then $a^n = a^n y$. It only remains to show that $r(a^n) = r(y)$. Let $x \in r(a^n)$, then $a^n x = 0$ and $x a^n a^n t = 0$, so $x a^{2n} t = 0$ and $x y = 0$. Hence $x \in \ell (y) = r(y)$ and $r(a^n) \subseteq r(y)$. On the other hand, let $z \in r(y)$, then $y z = 0$ and $a^{2n} r z = 0$, this implies that $a^n a^{2n} t z = 0$, so $a^n z = 0$, thus $z \in r(a^n)$ and $r(y) \subseteq r(a^n)$. Therefore, $r(a^n) = r(y)$ and $\mathcal{R}$ is a right semi $\pi$-regular ring in the same method we can easily show that $\mathcal{R}$ is a left semi $\pi$-regular.

A ring $\mathcal{R}$ is called $\pi$-biregular [9] if for any $a \in \mathcal{R}$, $\mathcal{R} a^n \mathcal{R}$ is generated by a central idempotent, for some positive integer $n$.

We begin by stating following lemma, which will be used in proof of our main result.
Lemma 2.8: A ring $R$ is $\pi$-biregular if and only if for every $a \in R$, $R a^n R \oplus r(a^n) = R$, for some positive integer $n$.

Proof: See [9]. ♦

Proposition 2.9: Let $R$ be a reduced ring. Then, $R$ is an $s\pi$WR-ring if and only if $R$ is a $\pi$-biregular ring.

Proof: Assume that $R$ is $\pi$-biregular ring. Then, by Lemma 2.8, $R = r(a^n) \oplus R a^n R$. Since $R$ is reduced, then $r(a^n) = r(a^{2n})$ for every positive integer $n$ and hence $R = r(a^{2n}) \oplus R a^n R$. In particular, $1 = b + t_1 a^{2n} t_2$, for some $t_1, t_2 \in R$ and $b \in r(a^n)$. So, $a^n = a^n t_1 a^{2n} t_2$. Therefore, $R$ is $s\pi$WR-ring.

Conversely, assume that $R$ is $s\pi$WR-ring, then for every $a \in R$, there exist $t_1, t_2 \in R$ such that $a^n = a^n t_1 a^{2n} t_2$, for some positive integer $n$. This implies that $a^n = a^n t_1 a^n t_2$. So, $a^n = a^n t_1 a^n t (t = a^n t_2)$, for some $t \in R$. Thus, $a^n R = a^n R a^n R$. Also, by [8, Theorem 2.7], $R = r(a^n) \oplus R a^{2n} R$. Thus, $R = r(a^n) \oplus R a^n R$. So, by Lemma 2.8, $R$ is $\pi$-biregular ring. ♦

3. Commutative $s\pi$-Weakly Regular Rings

In this section, all our rings are commutative. We discuss some properties of $s\pi$-weakly regular rings.

We now introduce the following lemma, which may be used frequently in the sequel.

Lemma 3.1: A ring $R$ is local if and only if for any elements $r, s \in R$ such that $r + s = 1$ implies that either $r$ or $s$ is a unit.

Proof: See [6]. ♦

Theorem 3.2: Let $R$ be a local, $s\pi$-weakly regular ring. Then, every element in $R$ is either a unit or nilpotent.

Proof: Assume that $R$ is a local ring and $a \in R$. Since $R$ is $s\pi$-weakly regular, there exist $x, y \in R$ such that $a^n = a^n x a^{2n} y$, for some positive integer $n$. So, $a^n (1 - x a^{2n} y) = 0$ and hence $a^n (1 - a^{2n} x y) = 0$. If $a^n = 0$, then $a$ is nilpotent. If $1 - a^{2n} x y \neq 0$, and $a^n \neq 0$, then $1 - a^{2n} x y$ is a zero divisor, that is $1 - a^{2n} x y$ is non unit. Since $(1 - a^{2n} x y) + a^{2n} x y = 1$ by Lemma 3.1,
a \, ^{2n} \, x \, y \, is \, a \, unit. \, This \, implies \, that \, a \, is \, a \, unit. \, If \, 1- \, a \, ^{2n} \, x \, y \, = \, 0, \, then \, a \, is \, a \, unit. \, Which \, completes \, the \, proof. \, •

**Proposition 3.3:**

Let R be \( s_{\pi} \)-weakly regular ring. Then, each element of R is either a unit or zero divisor.

**Proof:**

Let \( a \) be a non zero divisor in R and R is \( s_{\pi} \)-weakly regular, there exist elements \( x, \, y \in R \) and a positive integer \( n \) such that \( a^n = a^n \, x \, a^{2n} \, y \). So, \( a^n \, (1- \, x \, a^{2n} \, y) = 0 \) and hence \( a^n \, (1- \, a^{2n} \, x \, y) = 0 \). Since \( a \) is a non zero divisor, then \( a^n \) is nonzero divisor. Therefore, \( 1- \, a^{2n} \, x \, y \, = \, 0 \), which implies that \( a^{2n} \, (x \, y) \, = \, 1 \). Hence \( a^{2n} \, x \, y \, = \, a \, (a^{2n-1} \, (x \, y)) \, = \, 1 \). Thus, \( a \) is a unit. •

**Theorem 3.4:**

If \( P \) is a prime ideal of a ring \( R \) and \( R / P \) is \( s_{\pi} \)-weakly regular ring. Then, \( P \) is a maximal ideal.

**Proof:**

Let \( a \in R \). Then, \( a \, + \, P \in R / P \), since \( R / P \) is \( s_{\pi} \)-weakly regular, there exist \( b \, + \, P, \, c \, + \, P \in R / P \) and a positive integer \( n \) such that

\[
a^n \, + \, P \, = \, (a^n \, + \, P) \, (b \, + \, P) \, (a^{2n} \, + \, P) \, (c \, + \, P)
\]

\( a^n \, + \, P \, = \, a^n \, b \, a^{2n} \, c \, + \, P \). So, \( a^n - a^n \, b \, a^{2n} \, c \in P \) and hence \( a^n \, (1- \, b \, a^{2n} \, c) \in P \).

Since \( a^n \notin P \), then \( (1- \, b \, a^{2n} \, c) \in P \), gives

\[
1 \, + \, P \, = \, b \, c \, a^{2n-1} \, a \, + \, P
\]

\( = \, (b \, c \, a^{2n-1} \, + \, P) \, (a \, + \, P) \), this shows that \( a \, + \, P \) has an inverse. Thus, \( R/P \) is a division ring. Whence \( P \) is a maximal ideal. •

**Corollary 3.5:**

If R is an \( s_{\pi} \)-weakly regular ring, then every prime ideal of R is maximal.

**Proof:**

Let \( P \) be a prime ideal of R. Since R is \( s_{\pi} \)-weakly regular ring, then by Theorem 2.3, \( R/P \) is \( s_{\pi} \)-weakly regular. Thus, by Theorem 3.4, \( P \) is a maximal. •

**Lemma 3.6:**

If I is a primary ideal, then \( \sqrt{I} \) is prime.

**Proof:**

See [5]. •

**Lemma 3.7:**

Let I be an ideal of R. If \( \sqrt{I} \) is a maximal ideal of R, then I is primary ideal.
Proof:

See [2]. ♦

Theorem 3.8:

If \( R \) is \( s\pi \)-weakly regular ring and \( I \) is an ideal of \( R \), then \( I \) is primary if and only if \( \sqrt{I} \) is prime.

Proof:

If \( I \) is primary, then by Lemma 3.6, \( \sqrt{I} \) is prime.

Conversely, suppose that \( \sqrt{I} \) is prime. Since \( R \) is \( s\pi \)-weakly regular ring, then by Corollary 3.5, \( \sqrt{I} \) is maximal. Therefore, by Lemma 3.7, \( I \) is primary. ♦

The following result is the relationship between \( s\pi \)-weakly regular ring with its ideals by adding the condition that every ideal of \( R \) is completely semi-prime.

Theorem 3.9:

If every ideal of \( R \) is completely semi-prime, then \( R \) is \( s\pi \)-weakly regular if and only if for each ideal \( I \) of \( R \), \( I = \sqrt{I} \) holds.

Proof:

Let \( R \) be \( s\pi \)-weakly regular. It is obvious that \( I \subseteq \sqrt{I} \). Now, let \( b \in \sqrt{I} \), then \( b^n \in I \) and hence \( b^{2n} \in I \), for some positive integer \( n \). Now, \( b^n = b^n t_1 b^{2n} t_2 \), for some \( t_1, t_2 \in R \). Since \( b^n \in I \) and every ideal of \( R \) is completely semi-prime, then \( b \in I \). Therefore, \( \sqrt{I} = I \).

Conversely, assume that \( I = \sqrt{I} \), for each ideal \( I \) of \( R \). If we set \( I = a^n R a^{2n} R \), for some positive integer \( n \). Therefore, \( a^n R a^{2n} R = \sqrt{a^n R a^{2n} R} \). Now, \( a^{3n} \in a^n R a^{2n} R \), then \( a^n \in \sqrt{a^n R a^{2n} R} = a^n R a^{2n} R \). So, \( a^n \in a^n R a^{2n} R \). Whence \( R \) is \( s\pi \)-weakly regular. ♦
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